CHAPTER 6: SYSTEMS OF PARTICLES.
Dynamics of a RIGID BODY

Tipler / Mosca:	chapters 9 and 10
Ohanian:	chapters 12 and 13

Aritz Leonardo

eman ta zabal zazu

Universidad del País Vasco
Euskal Herriko Unibertsitatea

What is a RIGID BODY?

When a system of particles maintains constant (unchanged) the distance among all its particles, is said to be a rigid solid. (Of course such a solid does not exist, there is always deformation!)

But how will it move? are we able to describe its motion?

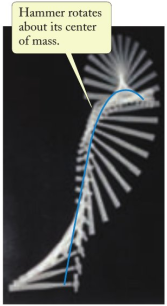
Remember that a RIGID BODY (RB) is a particular type of a PARTICLE SYSTEM. This means:

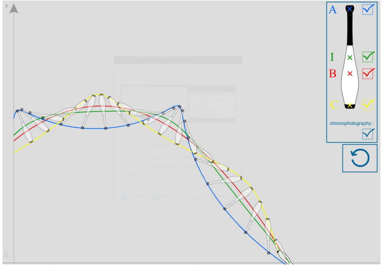
$\vec{F}_{\text{ext}} = M\vec{a}_{\text{CM}}$ is valid and applicable.

MOTION OF A RIGID BODY

- A rigid body can simultaneously have two kinds of motion: it can change its position in space (TRANSLATION), and it can change its orientation in space (ROTATION).
- In the general case of motion of a rigid body, the axis of rotation can have any direction and can also change its direction. To describe such complicated motion, it is convenient to separate the rotation into three components along three perpendicular axes but...

DON'T WORRY! we will never deal with the general case of rotation. All our rigid bodies will simply rotate about a fixed axis easily detectable which will be perpendicular to the paper.





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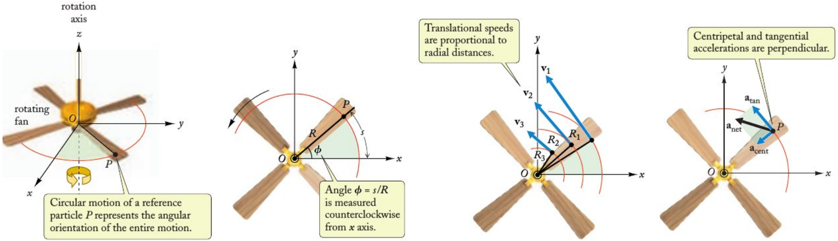
ROTATION ABOUT A FIXED AXE



$$\boxed{\phi(t) = \frac{s(t)}{R}}$$

$$\langle\omega\rangle = \frac{\Delta\phi}{\Delta t} \rightarrow \quad \omega = \frac{d\phi}{dt}$$

$$\langle\alpha\rangle = \frac{\Delta\omega}{\Delta t} \rightarrow \quad \alpha = \frac{d\omega}{dt}$$

$$f = \frac{\omega}{2\pi} \qquad T = \frac{1}{f}$$

$$s(t) = \phi(t)R$$

$$\frac{ds}{dt} = \frac{d\phi}{dt}R$$

$$\boxed{v = \omega R}$$

$$\frac{dv}{dt} = \frac{d\omega}{dt}R$$

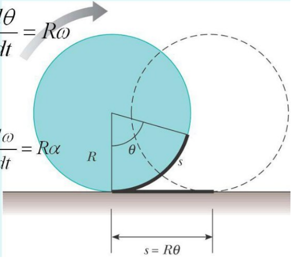
$$\boxed{a_T = \alpha R}$$

$$\boxed{a_C = \frac{v^2}{R}}$$

Rolling Motion

$$v_{CM} = \frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega$$

$$a_{CM} = \frac{dv_{CM}}{dt} = R\frac{d\omega}{dt} = R\alpha$$



KINETIC ENERGY OF A MOVING RIGID BODY

Remember ... MOVEMENT = TRANSLATION + ROTATION (fixed axis) The total kinetic energy of a rotating rigid body is simply the sum of the individual kinetic energies of all the particles.

$$E_k = \frac{1}{2} M v_{\mathrm{CM}}^2 + E_k^{'} \quad \text{energy of all the particles w.r.t. rotation axis}$$

$$E_k^{'} = \frac{1}{2} m_1 v_1^{'2} + \frac{1}{2} m_2 v_2^{'2} + \frac{1}{2} m_3 v_3^{'2} + \cdots$$

In a rigid body rotating about a given axis, all the particles move with the same angular velocity ω along circular paths:

$$v_1^{'} = R_1 \omega \quad v_2^{'} = R_2 \omega \quad v_3^{'} = R_3 \omega \quad \cdots$$

$$E_k^{'} = \frac{1}{2} m_1 R_1^2 \omega^2 + \frac{1}{2} m_2 R_2^2 \omega + \frac{1}{2} m_3 R_3^2 \omega + \cdots$$

$$E_k^{'} = \frac{1}{2} I \omega^2$$

$$\boxed{I \equiv m_1 R_1^2 + m_2 R_2^2 + m_3 R_3^2 + \cdots} \quad \text{MOMENT OF INERTIA}$$

The moment of inertia is a measure of the resistance that a body offers to changes in its rotational motion.

MOMENT OF INERTIA EXAMPLE: discrete mass

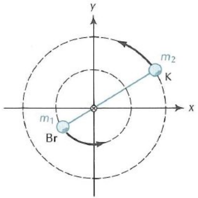


Fig. 12.11 The bromine and potassium atoms revolve about their common center of mass. The atoms can be regarded as particles joined by a rod.

EXAMPLE 5. Suppose that the molecule of potassium bromide (KBr) described in Example 10.5 rotates rigidly about its center of mass (Figure 12.11). What is the moment of inertia of the molecule about this axis? Suppose that the molecule rotates with an angular velocity of 1.0×10^{12} radian/s. What is the rotational kinetic energy?

SOLUTION: We can regard the molecule as a rigid body, consisting of two particles joined by a massless rod. It is best to place the origin of coordinates at the center of mass (see Figure 12.11). According to the results obtained in Example 10.5, the distance of the Br atom is then $R_1 = 0.93$ Å, and that of the K atom is $R_2 = 1.89$ Å. The corresponding masses are $m_1 = 79.9$ u and $m_2 = 39.1$ u. The moment of inertia of the molecule is

$$I = m_1 R_1^2 + m_2 R_2^2$$

$$= 79.9 \text{ u} \times (0.93 \text{ Å})^2 + 39.1 \text{ u} \times (1.89 \text{ Å})^2$$

$$= 2.09 \times 10^2 \text{ u} \cdot \text{Å}^2$$

$$= 2.09 \times 10^2 \text{ u} \cdot \text{Å}^2 \times \frac{1.66 \times 10^{-27} \text{ kg}}{1 \text{ u}} \times \left(\frac{10^{-10} \text{ m}}{1 \text{ Å}}\right)^2$$

$$= 3.47 \times 10^{-45} \text{ kg} \cdot \text{m}^2$$

The kinetic energy is

$$K = \tfrac{1}{2} I \omega^2$$

$$= \tfrac{1}{2} \times 3.47 \times 10^{-45} \text{ kg} \cdot \text{m}^2 \times (1.0 \times 10^{12} \text{ radians/s})^2$$

$$= 1.73 \times 10^{-21} \text{ J}$$

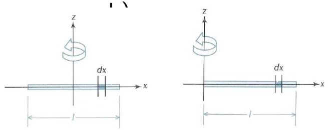
MOMENT OF INERTIA EXAMPLE: continuous mass

Consider a macroscopic solid body with its mass distributed throughout a volume. We can calculate the moment of inertia by subdividing the body into small mass elements and adding each contribution. In the limit $\Delta m_i \to 0$, this approximation becomes exact, and the sum becomes an integral.

$$I = \sum_{i=1}^{n} m_i R_i^2 \Rightarrow \boxed{I = \int_V R^2 \, dm}$$

Rod:

a)

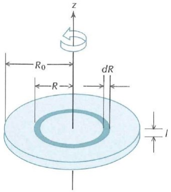


$$\lambda = \frac{M}{L} = \frac{dm}{dx}$$

$$\text{a)} \quad I = \int_{-L/2}^{L/2} x^2 \lambda dx = \frac{1}{12} M L^2$$

$$\text{b)} \quad I = \int_{0}^{L} x^2 \lambda dx = \frac{1}{3} M L^2$$

Disk:

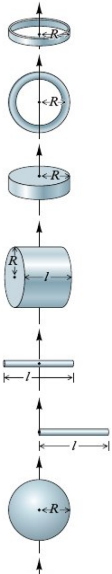


$$\sigma = \frac{M}{\pi R_0^2} = \frac{dm}{2\pi R dR}$$

$$I = \int_0^{R_0} R^2 \sigma 2\pi R dR = \frac{1}{2} M R_0^2$$

PARALLEL-AXIS THEOREM: Steiner

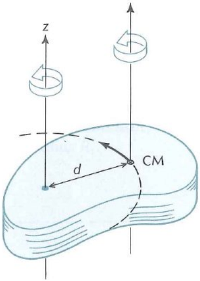
Calculated moments of inertia table:



BODY	MOMENT OF INERTIA
Thin hoop about symmetry axis	MR^2
Thin hoop about diameter	$\frac{1}{2}MR^2$
Disk or cylinder about symmetery axis	$\frac{1}{2}MR^2$
Cylinder about diameter through center	$\frac{1}{4}MR^2 + \frac{1}{12}Ml^2$
Thin rod about perpendicular axis through center	$\frac{1}{12}Ml^2$
Thin rod about perpendicular axis through end	$\frac{1}{3}Ml^2$
Sphere about diameter	$\frac{2}{5}MR^2$

Without prove we state the parallel-axis theorem that relates the moment of inertia I_{CM} about an axis through the CM to any other moment of inertia I about a parallel axis.

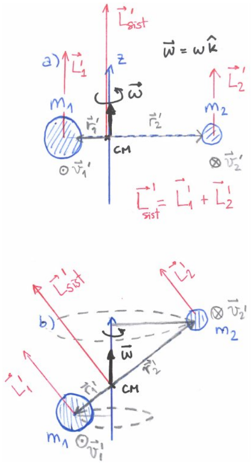
$$\boxed{I = I_{\mathrm{CM}} + Md^2}$$



ANGULAR MOMENTUM OF A RIGID BODY

In the previous chapter we have defined the total angular momentum of a system and its relation with an observer placed at the CM:

$$\vec{L}_{\text{syst}} = \sum_i m_i \vec{r}_i \times \vec{v}_i \;\Rightarrow\; \vec{L}_{\text{syst}} = M\vec{r}_{\text{CM}} \times \vec{v}_{\text{CM}} + \vec{L}'_{\text{syst}} \quad \left(\vec{L}'_{\text{sist}} = \sum_i m_i \vec{r}'_i \times \vec{v}'_i\right)$$



System = RIGID BODY = RB

$$|\vec{L}'_{\text{RB}}| = |\vec{L}'_1| + |\vec{L}'_2| = m_1 r'_1 v'_1 \sin 90 + m_2 r'_2 v'_2 \sin 90$$

In a rigid body: $v'_1 = \omega r'_1$ and $v'_2 = \omega r'_2$

$$|\vec{L}'_{\text{RB}}| = m_1 r'^2_1 \omega + m_2 r'^2_2 \omega = \underbrace{\left(m_1 r'^2_1 + m_2 r'^2_2\right)}_{I \equiv \text{moment of inertia}} \omega$$

$$\boxed{\vec{L}'_{\text{RB}} = I\vec{\omega}}$$

DYNAMICS OF A RIGID BODY: EQUATION OF MOTION

We arrive at the main equations of motions which are useful to describe the movement of the RB. For particle systems we demonstrated that the total torque on a system ($\vec{\tau}$) due to external forces was equal to the rate of change of angular momentum, of course it will hold for RB:

$$\boxed{\frac{d\vec{L}'_{\text{RB}}}{dt} = \vec{\tau}'_{\text{RB}} \quad \text{external forces only!} \quad \left(\sum_i \vec{\tau}_i = \vec{\tau}'_{\text{RB}} = 0 \rightarrow \vec{L}' = \text{constant}\right)}$$

$$\frac{d\vec{L}'_{\text{RB}}}{dt} = \frac{d}{dt}(I\vec{\omega}) = I\frac{d\vec{\omega}}{dt} = I\vec{\alpha} = \vec{\tau}'_{\text{RB}}$$

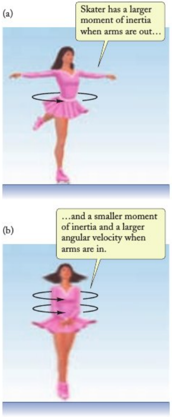
a) $\boxed{\vec{\tau}'_{\text{RB}} = I\vec{\alpha}}$

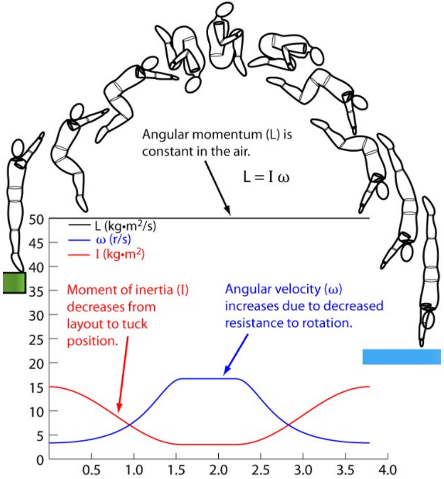
b) $\boxed{\vec{F}^{\text{RB}}_{\text{ext}} = M\vec{a}_{\text{CM}}}$

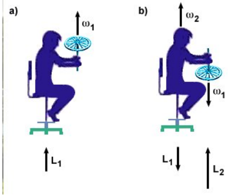
CONSERVATION OF THE ANGULAR MOMENTUM

Suppose there are no external forces and thus the external torque is zero. So...

$$\boxed{\vec{L} = \text{constant} = I\vec{\omega}}$$







WORK AND ENERGY OF THE RIGID BODY

We again recover an observation made for the particle system:

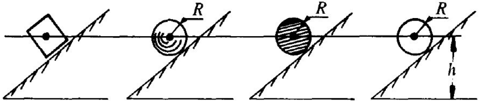
$$W^{\text{total}}_{\text{external}} + W^{\text{total}}_{\text{internal}} = E^{\text{syst}}_{k,B} - E^{\text{syst}}_{k,A}$$

By the definition all particles within a rigid body must keep their mutual distances constant, so this implies $\Rightarrow W^{\text{total}}_{\text{internal}} = 0$

$$\boxed{W^{\text{total}}_{\text{external}} = E^{\text{syst}}_{k,B} - E^{\text{syst}}_{k,A}}$$

In other words, if the external forces acting upon our RB are conservative (as gravity, spring ...) the Energy will be conserved and we can write $E_{\text{A}} = E_{\text{B}}$

Example: A cube, an sphere, a cylinder and ring of equal radius and mass are thrown downhill from a height h. What will be the order of arrival?



$$v^{\text{cube}}_{\text{CM}} = \sqrt{2gh} \qquad v^{\text{sphere}}_{\text{CM}} = \sqrt{\frac{10}{7}gh} \qquad v^{\text{cylinder}}_{\text{CM}} = \sqrt{\frac{4}{3}gh} \qquad v^{\text{ring}}_{\text{CM}} = \sqrt{gh}$$