7. Gaia: Elektrostatika

Tipler eta Mosca:	21. eta 22. kapituluak
Fisika Orokorra:	18. kapitulua
Ohanian:	23. eta 24. kapituluak

Aritz Leonardo

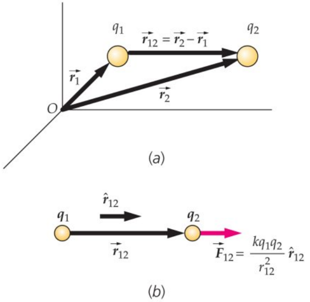
eman ta zabal zazu

Universidad del País Vasco
Euskal Herriko Unibertsitatea

Sarrera

Coulomb's law

The magnitude of the electric force that a particle exerts on another particle is directly proportional to the product of their charges and inversely proportional to the square of the distance between them. The direction of the force is along the line joining the particles.



$$\boxed{\vec{F} = K\frac{q_1 q_2}{r_{12}^2}\hat{r}_{12}}$$

$$k = 8.99 \times 10^9 \text{Nm}^2/\text{C}^2$$

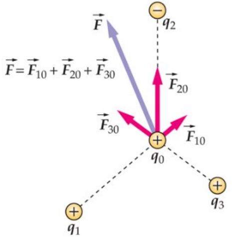
$$k = \frac{1}{4\pi\varepsilon_0} \text{ where}$$

$$\varepsilon_0 = 8.85 \times 10^{-12} \text{C}^2/(\text{Nm}^2)$$

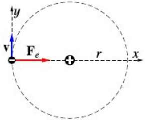
This equation applies to particles (electrons and protons) and also to any small charged bodies, provided that the sizes of these bodies are much less than the distance between them: such bodies are called point charges.

Coulomb's law II

When there are multiple charges, the superposition principle applies. Total force over charge 0 exerted by all the other is: $\vec{F}_T^0 = \vec{F}_{10} + \vec{F}_{20} + \ldots$



The hydrogen atom: $m_e = 9.1 \times 10^{-31}$kg, $r = 0.53 \times 10^{-10}$m
$q_e = -1.6 \times 10^{-19}$C



$$F_e = F_n \quad \rightarrow \quad \frac{kq^2}{r^2} = \frac{mv^2}{r} \quad \rightarrow \quad v = q\sqrt{\frac{k}{mr}} = \boxed{2.25 \times 10^6 \text{ m/s} = v}$$

Exercises 1, 3, 4 and 17

Electric field

To discover the field at a given position, take a (virtual) point charge q (a "test charge") and place it at that position. The charge q will then feel an electric force $\vec{F}$; The electric field $\vec{E}$ is defined as the force $\vec{F}$ divided by the magnitude of the charge q. In other words, the electric field is simply the force per unit positive charge.

$$\boxed{\vec{E} = \frac{\vec{F}}{q}}$$

The electric field that a point charge Q generates a distance r is:

$$\boxed{\vec{E} = K\frac{Q}{r^2}\hat{r}}$$

The superposition principle also applies to the electric field.

$$\vec{E}_P = \sum_i \vec{E}_{iP} = \sum_{i=1}^{N} k\frac{Q_i}{r_i^2}\hat{r}_i$$

Exercises 21, 22

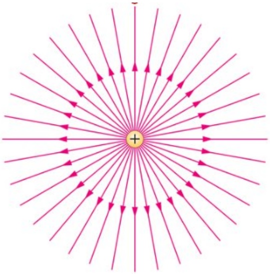
Lines of Electric Field

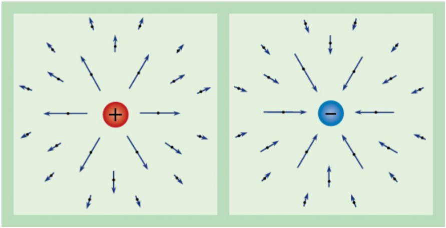
The electric field can be represented graphically by drawing, at any given point of space, a vector whose magnitude and direction are those of the electric field at that point.

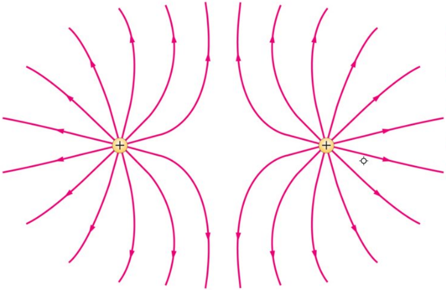
Alternatively, the electric field can be represented graphically by field lines:

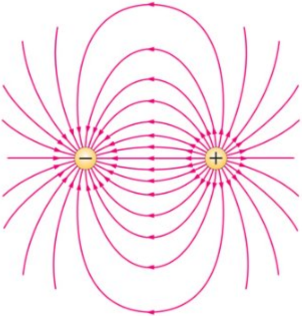
- At any given point, this lines are drawn tangent to the electric field.
- The density of the lines is directly proportionalto the magnitude of the field.
- We begin each line on a positive point and end on a negative point charge.
- Field lines never intersec.
- They are NOT trajectories that a charged particle would follow if left at rest a certain point!

Lines of Electric Field II

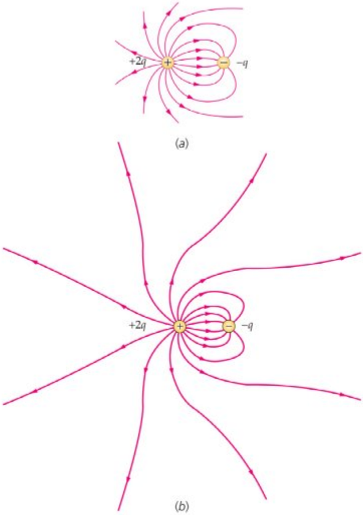








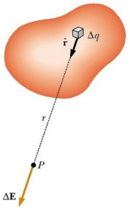
Lines of Electric Field II

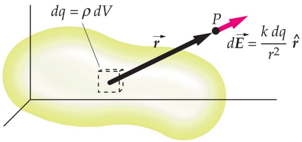


Electric Field II: Macroscopic charged bodies, continuous charge distributions.

In a microscopic scale, electric charge is quantized. However, there are often situations in which many charges are so closed together that they can be thought of as continuously distributed. In both cases, it is usually easy to find a volume element V that is large enough to contain a multitude of individual charges yet is small enough that replacing V with a differential dV and using calculus introduces negligible error. The charge density concept is the key ingredient to jump from discrete to continuous configurations.

$$\boxed{\rho = \frac{\Delta Q}{\Delta V} \qquad \sigma = \frac{\Delta Q}{\Delta A} \qquad \lambda = \frac{\Delta Q}{\Delta L}}$$

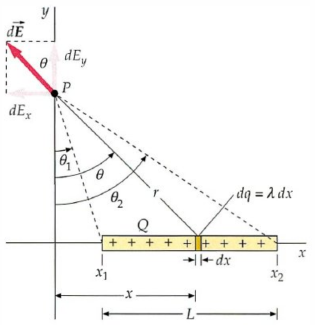




$$\vec{E} = \lim_{N\to\infty} \sum_{i=1}^{N} k\frac{\Delta Q_i}{r_i^2}\hat{r}_i = \int k\frac{dq}{r^2}\hat{r} \quad \Rightarrow \quad \boxed{\vec{E} = \int dE\,\hat{r} = \int k\frac{dq}{r^2}\hat{r}}$$

Electric Field II: continuous charge distributions. EXAMPLES

1) A charge Q is uniformly distributed on a straight-line segment of length L:



$$\lambda = Q/L \rightarrow dq = \lambda dx$$

$$|d\vec{E}| = \frac{kdq}{r^2} = \frac{k\lambda dx}{r^2}$$

$$d\vec{E} = -\, dE_x\hat{\mathrm{i}} + dE_y\hat{\mathrm{j}}$$

$$dE_y = |d\vec{E}| \cos\theta = \frac{k\lambda dx}{r^2}\frac{y}{r}$$

$$(\text{where: } \cos\theta = \frac{y}{r} \quad \text{and} \quad r = \sqrt{x^2 + y^2})$$

$$E_y = \int_{x=x_1}^{x=x_2} dE_y = k\lambda y \int_{x_1}^{x_2} \frac{dx}{r^3}$$

From the figure we can see that $x = y\tan\theta$ so $dx = y\frac{d\theta}{\cos^2\theta}$, $y = r\cos\theta$

$$E_y = k\lambda y\frac{1}{y^2}\int_{\theta_1}^{\theta_2} \cos\theta d\theta = \frac{k\lambda}{y}(\sin\theta_2 - \sin\theta_1) = \boxed{\frac{kQ}{Ly}(\sin\theta_2 - \sin\theta_1)}$$

Show that $E_x = \frac{k\lambda}{y}(\cos\theta_2 - \cos\theta_1)$

Electric Field II: continuous charge distributions. EXAMPLES

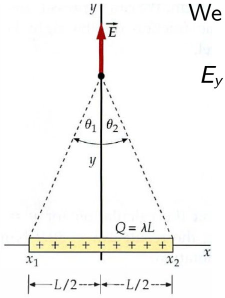
2) $\vec{E}$ due to an infinite line charge.

A line charge may be considered infinite if for any field point of interest P $x_1 \to -\infty$ and $x_2 \to +\infty$ or $\theta_1 \to -\pi/2$ and $\theta_2 \to +\pi/2$

$$E_x = 0 \qquad E_y = \frac{2k\lambda}{y} \qquad (E_R = 2k\frac{\lambda}{R})$$

3) Electric field on the axis of a finite line charge

$x_1 = -\frac{1}{2}L$ and $x_2 = \frac{1}{2}L$ so $\theta_1 = -\theta_2$



We recover result 1) and substitute the particular values.

$$E_y = \frac{k\lambda}{y}(\sin\theta_2 - \sin\theta_1) = \frac{k\lambda}{y}[\sin\theta - \sin(-\theta)] = \frac{2k\lambda}{y}\sin\theta$$

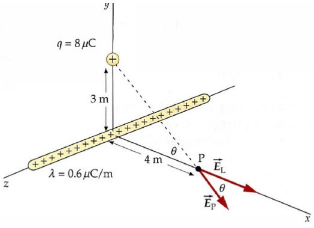
$$\sin\theta = \frac{\frac{1}{2}L}{\sqrt{(\frac{1}{2}L)^2 + y^2}}$$

$$E_x = \frac{k\lambda}{y}(\cos\theta_2 - \cos\theta_1) = \frac{k\lambda}{y}[\cos\theta - \cos(-\theta)] = 0$$

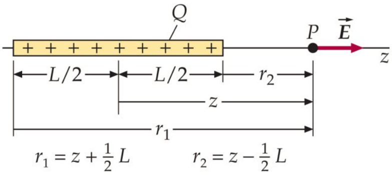
$$\vec{E} = E_x\hat{\imath} + E_y\hat{\jmath} = \boxed{\frac{2k\lambda}{y}\frac{\frac{1}{2}L}{\sqrt{(\frac{1}{2}L)^2 + y^2}}\hat{\jmath}}$$

Electric Field II: continuous charge distributions. EXAMPLES

Exercise 1: Calculate the electric field at point P of the picture

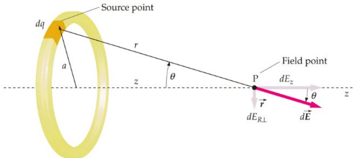


Exercise 2: Calculate the electric field at point P of the picture

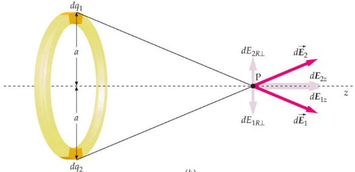


Electric Field II: continuous charge distributions. EXAMPLES

4) $\vec{E}$ on the axis of a charged ring



(a)



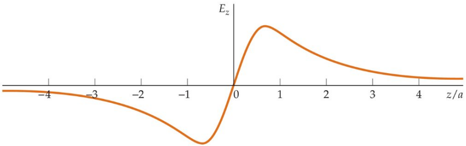
(b)

$$dE_z = \frac{kdq}{r^2}\cos\theta = \frac{kdq}{r^2}\frac{z}{r}$$

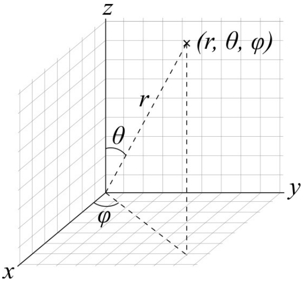
where $r^2 = z^2 + a^2$

and $\cos\theta = \frac{z}{r} = \frac{z}{\sqrt{z^2 + a^2}}$

$$E_z = \int \frac{kzdq}{(z^2 + a^2)^{3/2}} = \frac{kz}{(z^2 + a^2)^{3/2}} \int dq = \boxed{\frac{kzQ}{(z^2 + a^2)^{3/2}}}$$



Coordinate systems



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$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \arccos \frac{z}{\sqrt{x^2 + y^2 + z^2}} = \arccos \frac{z}{r}$$

$$\varphi = \arctan \frac{y}{x}$$

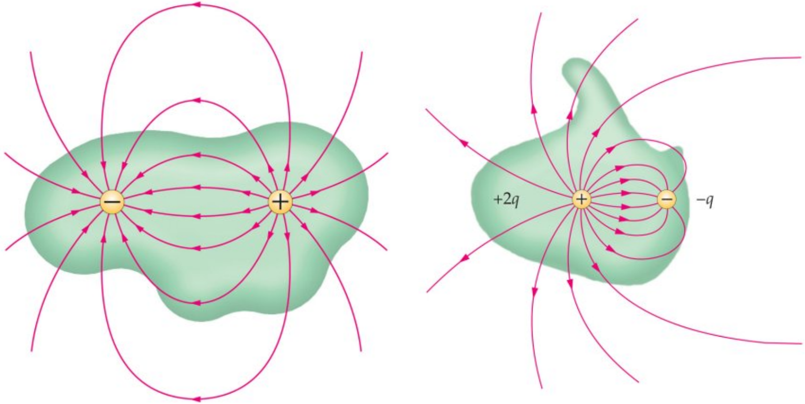
$$x = r \sin\theta \cos\varphi$$

$$y = r \sin\theta \sin\varphi$$

$$z = r \cos\theta$$

Gauss's law

A closed surface is one that divides the universe into two distinct regions, the region inside the surface and the region outside the surface. FLUX: Is to count the net number of lines OUT of the surface. Count +1 if the line penetrates from inside and -1 when it penetrates from the outside.

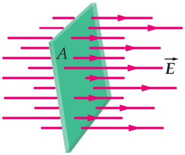


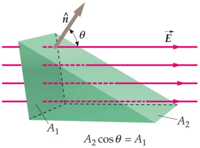
The net number of lines out of any surface enclosing the charges is proportional to the net charge enclosed by the surface. This rule is a qualitative statement of Gauss's law

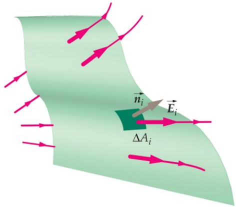
Gauss's law

The mathematical quantity that corresponds to the number of field lines pene trating a surface is called the electric flux ϕ.

$$\phi = EA \qquad \phi = \vec{E} \cdot \hat{n} A \qquad \phi = \lim_{\Delta A_i \to 0} \sum_i \vec{E}_i \cdot \Delta A_i = \int_S \vec{E} \cdot \hat{n} dA$$





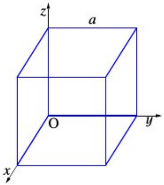


GAUSS'S LAW $\boxed{\phi_{net} = \oint_S \vec{E} \cdot \hat{n} dA = \frac{Q_{\text{inside}}}{\epsilon_0}}$

Gauss's law is valid for all surfaces and all charge distributions. For charge distributions that have high degrees of symmetry, it can be used to calculate the electric field! Check the following examples

Gauss's law

Example 1: If $\vec{E} = 9x\hat{\mathrm{i}}$ N/C calculate the net flux out of the cube of side a

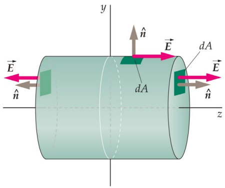


$$\phi = 9a\int_{x=a} \hat{\mathrm{i}}\cdot dA\hat{\mathrm{i}} + 9\cancel{\int_{x=0} 0\hat{\mathrm{i}}\cdot(-dA)\hat{\mathrm{i}}} + 9\cancel{\int_{y=a} x\hat{\mathrm{i}}\cdot(dA)\hat{\mathrm{j}}}$$

$$+ 9\cancel{\int_{y=0} 0\hat{\mathrm{i}}\cdot(-dA)\hat{\mathrm{j}}} + 9\cancel{\int_{z=a} x\hat{\mathrm{i}}\cdot(dA)\hat{\mathrm{k}}} + 9\cancel{\int_{z=a} x\hat{\mathrm{i}}\cdot(-dA)\hat{\mathrm{k}}}$$

$$= 9a\int dA = 9a^3 \text{ Wb} \neq 0$$ Exists net charge inside the cube!

Example 2: An electric field is $\vec{E} = (200N/C)\hat{\mathrm{k}}$ in the region $z > 0$ and $\vec{E} = (-200N/C)\hat{\mathrm{k}}$ in the region $z < 0$. a) What is the net outward flux through the entire closed cylindrical surface of the picture? b) What is the net charge inside the closed surface? (The Cylinder has R=5cm and L=20cm)



$$\phi_{\text{right}} = \vec{E}_{\text{right}}\cdot\hat{\mathrm{k}}\pi R^2 = 200\hat{\mathrm{k}}\cdot\hat{\mathrm{k}}\pi 0.05^2 = 1.57\text{Wb}$$

$$\phi_{\text{left}} = \vec{E}_{\text{left}}\cdot(-\hat{\mathrm{k}})\pi R^2 = 200(-\hat{\mathrm{k}})(-\hat{\mathrm{k}})\pi 0.05^2 = 1.57\text{Wb}$$

$$\phi_{\text{curved}} = 0$$

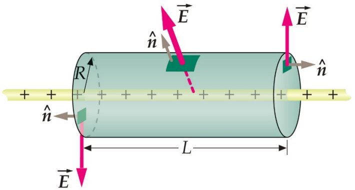
$$\phi_{\text{net}} = \phi_{\text{right}} + \phi_{\text{left}} + \phi_{\text{curved}} = 3.14\text{Wb}$$

$$Q_{\text{inside}} = \epsilon_0\phi_{\text{net}} = (8.85\times 10^{-12}\text{C}^2/\text{Nm}^2)(3.14\times \text{Nm}^2/C) = 2.78\times 10^{-11}C$$

Gauss's law: A way to calculate electric fields

Given a highly symmetrical charge distribution, the electric field can be calculated more easily using Gauss's law than it can be using Coulomb's law. We first find an imaginary closed surface, called a Gaussian surface. This surface is chosen so that on each of its pieces $\vec{E}$ is either 0, perpendicular to $\hat{n}$, or parallel to $\hat{n}$ with $\vec{E}$ constant. Then the flux through each piece equals $E_n A$ and Gauss's law is used to relate $\vec{E}$ to the charges inside the closed surface.

Example 1: The infinite line charge revisited. So easy!

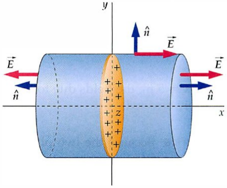


$$\begin{aligned}
\phi_{\text{curved}} &= \vec{E}\cdot\hat{n}A_{\text{curved}} = \vec{E}\cdot\hat{R}A_{\text{curved}} \\
&= E_R 2\pi RL \\
\phi_{\text{left}} &= \vec{E}\cdot\hat{n}A_{\text{left}} = 0 \\
\phi_{\text{right}} &= \vec{E}\cdot\hat{n}A_{\text{right}} = 0 \\
\phi_{\text{net}} &= \frac{Q_{\text{inside}}}{\epsilon_0} \Rightarrow E_R 2\pi RL = \frac{\lambda L}{\epsilon_0}
\end{aligned}$$

$$\boxed{\vec{E}_R = \frac{\lambda}{2\pi\epsilon_0 R}\hat{r}}$$

Gauss's law: A way to calculate electric fields

Example 2: The infinite charged plane

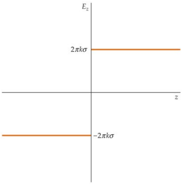


$$\phi_{net} = \oint_S \vec{E}\cdot\hat{n}dA = \frac{Q_{\text{inside}}}{\epsilon_0}$$

$$= \int_{\text{left}} E(-\hat{\text{i}})(-\hat{\text{i}})dA + \int_{\text{right}} E(\hat{\text{i}})(\hat{\text{i}})dA$$

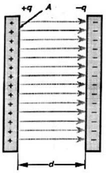
$$+ \cancel{\int_{\text{curve}} E(\pm\hat{\text{i}})(\hat{n})dA} = 2E\int dA = 2EA$$

$$2EA = \frac{Q_{\text{inside}}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0} \Rightarrow \qquad \boxed{E = \frac{\sigma}{2\epsilon_0}}$$

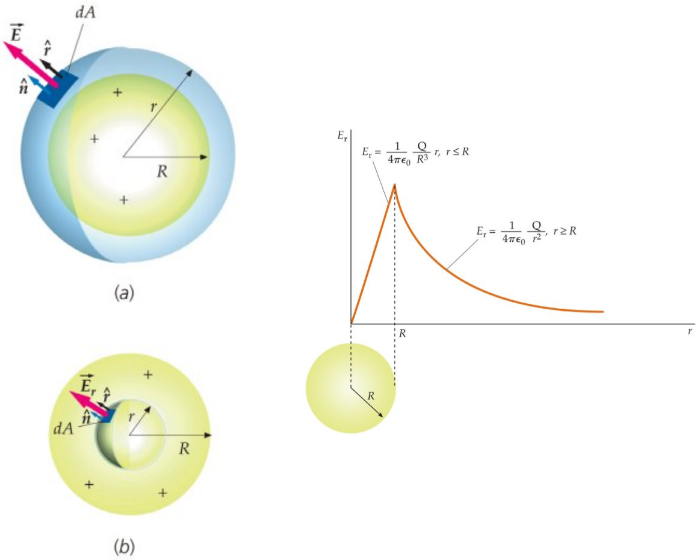


E is independent of the distance!

Example 3: The capacitor. Two planes in front of each other with opposite σ.



Gauss's law: A way to calculate electric fields



Electrostatic energy (U)

Imagine there is a fixed charge $+Q$ at a given position and I want to bring a second charge $+q$ from very far away (from ∞) up to a distance R close to it. The question is: How much energy do I have to spend? (imagine you travel radially)

$$W_{AL} = \int_{\infty}^{R} \vec{F}_{AL} \cdot d\vec{r} = -\int_{\infty}^{R} \vec{F}_{el} \cdot d\vec{r} = \int_{R}^{\infty} \vec{F}_{el} \cdot d\vec{r} =$$

$$= \int_{R}^{\infty} \frac{q_1 q_2}{4\pi\epsilon_0 r^2} dr = \frac{q_1 q_2}{4\pi\epsilon_0} \int_{R}^{\infty} \frac{dr}{r^2} = \frac{q_1 q_2}{4\pi\varepsilon_0} \left(\frac{-1}{r}\right)_{R}^{\infty} = \frac{q_1 q_2}{4\pi\varepsilon_0 R} \text{ [J]=U}$$

$$q_1 > 0 \text{ and } q_2 > 0 \text{ or } q_1 < 0 \text{ and } q_2 < 0 \Rightarrow U > 0$$

$$q_1 > 0 \text{ and } q_2 < 0 \text{ or } q_1 < 0 \text{ and } q_2 > 0 \Rightarrow U < 0$$

Although we have asked to travel radially, the result is completely general as the electric force is conservative. In other words, the result of the integral is independent of the path followed only depending on the initial and final points. Imagine you want to gather N charges close to each other, how much energy would you spend? $U = \frac{1}{2} \sum_{ij}^{N} \frac{q_i q_j}{4\pi\varepsilon_0 r_{ij}}$

Electrostatic potential (V)

In the same way we introduced earlier the concept of electric field, we are now going to arrive at the definition of the electrostatic potential by asking ourselves, what would be the energy cost PER CHARGE UNIT, when we bring our test-charge q up to a point "p" which is at a distance R from Q. (i.e. approximate our virtual +1C up to a distance R)

$$\boxed{V_p = \frac{U}{q} = \frac{Q}{4\pi\epsilon_0 R} \quad \text{J/C=volt}}$$

Note that "p" can be any point of the space, there are no restrictions, so we say that the potential at point p produced by Q is ...(see above)

If there are multiple point charges producing a potential at a point "p" we have to add each contribution:

$$\boxed{V_p = \frac{Q_1}{4\pi\epsilon_0 r_1} + \frac{Q_2}{4\pi\epsilon_0 r_2} + \cdots}$$

Electrostatic potential (V)

So far so good, we have defined the electrostatic potential V at "p" but note that the expression is ONLY valid when it is produced by point charges! Let's generalize this definition.

Imagine we want to move our virtual +1C from point a to point b, the potential difference will be the energy cost per unit charge:

$$\frac{U}{q} = -\int_a^b \frac{\vec{F}_{el}}{q} d\vec{l} = \boxed{-\int_a^b \vec{E} d\vec{l} = \Delta V = V_b - V_a}$$

We may have a very difficult charge distribution, but if by luck, someone provides me with the electric field ($\vec{E}$) that this distribution produces everywhere in space, I can integrate it and automatically know the electrostatic potential V. The opposite is also true!

$$\boxed{\vec{E} = -\vec{\nabla} V = -\frac{dV}{dx}\hat{\mathrm{i}} - \frac{dV}{dy}\hat{\mathrm{j}} - \frac{dV}{dz}\hat{\mathrm{k}}}$$

If the charge distribution originating the field has spherical or cylindrical symmetry it is more convenient to use the gradient in those coordinates:

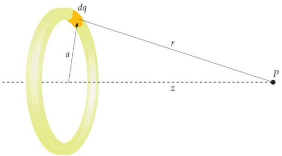
$$\boxed{\vec{E} = -\vec{\nabla} V = -\frac{dV}{dr}\hat{r}}$$

Electrostatic potential (V): Charge distributions

As we have done before, we can divide the charge distribution into differential pieces (dq) and add each contribution at the point of interest:

$$\boxed{V = \int \frac{kdq}{r}}$$

Example:



$$V = \int_0^Q \frac{kdq}{r} = \frac{k}{r}\int_0^Q dq = \frac{kQ}{r} = \frac{kQ}{\sqrt{z^2 + a^2}}$$

We can obtain $\vec{E}$ again but this time from the potential.

$$\vec{E} = -\cancel{\frac{dV}{dx}}\hat{\mathrm{i}} - \cancel{\frac{dV}{dy}}\hat{\mathrm{j}} - \frac{dV}{dz}\hat{\mathrm{k}} = \frac{kQz}{(z^2 + a^2)^{3/2}}\hat{\mathrm{k}} \quad \text{ok!}$$

Energy of a particle moving through an external field

Imagine a charge q moving through an external field (i.e. not generated by q). If we ignore gravitational interactions, the energy of the charge will be:

$$E = E_k + U = \frac{1}{2}mv^2 + qV$$

Exactly in the same way as we did for the gravitational force, electric force is also conservative thus, the total energy will remain constant:

$$E_1 = E_2 \Rightarrow \frac{1}{2}mv_1^2 + qV_1 = \frac{1}{2}mv_2^2 + qV_2$$

The total work performed by the electric field will thus be:

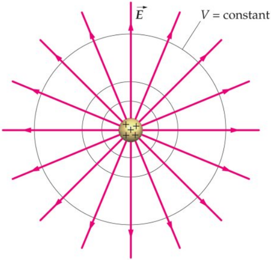
$$W = \Delta E_k = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 = q(V_1 - V_2)$$

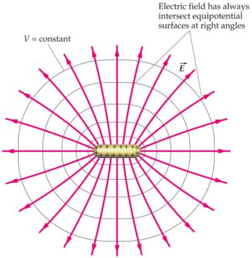
Example: Imagine that the ring of the previous slide has a radius of 4cm and carries a uniform charge of 8nC. A small particle of mass $m = 6$ mg and charge $q_0 = 5$ nC is placed at $z = 3$ cm and released. Find the speed of the particle when it is at a great distance from the ring.

$$E = kte \Rightarrow E_k^{\text{has}} + U^{\text{has}} = E_k^{\text{buk}} + U^{\text{buk}} \Rightarrow 0 + q_0 V(3) = E_z + q_0 \cancel{V(\infty)}$$

$$\frac{1}{2}mv_{\text{buk}}^2 = q_0 \frac{kQ}{\sqrt{0.03^2 + a^2}} \Rightarrow v_{\text{buk}} = 1.55 \text{ m/s}$$

Electrostatic potential (V): Equipotential Surfaces





An equipotential surface is a fictitious surface that joins all the points at equal potential. In other words, if I move following one of those surfaces the value of the potential keeps unchanged.

$$0 = dV = \vec{E} \cdot d\vec{l} \Rightarrow \vec{E} \perp d\vec{l}$$

Electric field lines must be perpendicular to equipotential surfaces